

Self-Similar Factor Approximants

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Abstract

The problem of reconstructing functions from their asymptotic expansions in powers of a small variable is addressed by deriving a novel type of approximants. The derivation is based on the self-similar approximation theory, which presents the passage from one approximant to another as the motion realized by a dynamical system with the property of group self-similarity. The derived approximants, because of their form, are named the self-similar factor approximants. These complement the obtained earlier self-similar exponential approximants and self-similar root approximants. The specific feature of the self-similar factor approximants is that their control functions, providing convergence of the computational algorithm, are completely defined from the accuracy-through-order conditions. These approximants contain the Padé approximants as a particular case, and in some limit they can be reduced to the self-similar exponential approximants previously introduced by two of us. It is proved that the self-similar factor approximants are able to reproduce exactly a wide class of functions which include a variety of transcendental functions. For other functions, not pertaining to this exactly reproducible class, the factor approximants provide very accurate approximations, whose accuracy surpasses significantly that of the most accurate Padé approximants. This is illustrated by a number of examples showing the generality and accuracy of the factor approximants even when conventional techniques meet serious difficulties.

1 Introduction

The problem of reconstructing functions from their perturbative asymptotic expansions in powers of a parameter or a variable is so frequently met in physics and in applied sciences that there is no necessity to explain its importance. The best known methods for such a reconstruction are the Padé approximation and Borel summation, including their variants and combinations [1,2]. These techniques usually require that a large number of terms of an asymptotic expansion be available. The Borel summation demands, in addition, that the high-order expansion coefficients be given and the analytic properties of the sought function on the complex plane be prescribed. However, the overwhelming majority of realistic physical problems are too complicated and perturbation theory is only able to derive a few first terms. And the luxury of knowing in advance the analytic properties of an unknown function, together with its high-order expansion coefficients, as is required for the Borel summation, is practically never available. Because of the latter, Padé approximants are more often employed in applications, although their usage also confronts with several difficulties, among which the most notorious are the appearance of spurious poles and the poor recovery of non-integer critical exponents.

An alternative approach to the problem of reconstructing functions has been developed, whose basic ideas are as follows. First of all, to improve the convergence property of a perturbative sequence, it is necessary to introduce *control functions* defined by an optimization procedure [3–5]. This idea makes the foundation of the *optimized perturbation theory* that is now widely employed for various applications [3–16]. The second pivotal idea is to consider the successive passage from one approximation to the next one as a dynamical evolution on the manifold of approximants, which is formalized by the notion of *group self-similarity* [17–22]. And the third principal point is the introduction of control functions in the course of rearranging perturbative asymptotic expansions by means of *algebraic transforms* [23–27]. Because of their specific scaling properties typical of fractals [28,29], the algebraic transforms can also be called *fractal transforms* [30]. By using this technique, two types of approximants have been obtained, *self-similar exponentials* and *self-similar roots* [24–27].

In the present paper, we suggest a different approximation scheme resulting in what may be named *self-similar factor approximants*. These new approximants possess an important principal property distinguishing them from the self-similar approximants mentioned above: the control parameters, entering the self-similar factors, can be completely defined from a given asymptotic expansion by the so-called “accuracy-through-order” matching method. This is in contrast with the self-similar exponentials whose controls, designed to improve convergence, are defined from additional optimization conditions. This method of accuracy-through-order is also different from the determination of the control parameters of the self-similar roots which are determined by matching two asymptotic expansions valid in the neighborhood of two different asymptotic points. Being based on the sole initial asymptotic expansion, the self-similar factors have the advantage of simplicity which makes their usage quite attractive. Furthermore, these approximants allow one to reconstruct *exactly* a wide class of functions. And when they do not yield exact answers, they provide very accurate approximations, essentially more accurate than given by Padé approximants.

We, first, give in the next section the mathematical foundation of the self-similar factor

approximants. Sections 3 and 4 are devoted to examples chosen both for their illustrative properties, the difficulties they pose to the more conventional Padé technique and the relevance to several physical problems (polymers, state- and velocity-dependent solid friction dynamics, critical phenomena in field theory and in Ising models).

2 Mathematical Foundation

2.1 Derivation of factor approximants

Assume that we are solving a complicated problem, aiming at finding a real function $f(x)$ of a real variable x . Because of the complexity of the problem, the only thing we are able to do is to invoke a kind of perturbation theory for obtaining approximate expressions $f_k(x)$ of order $k = 0, 1, 2, \dots$, valid in the asymptotic vicinity of $x = 0$. Usually, the asymptotic approximants $f_k(x)$ can be presented as a power series of x and written in the form

$$f_k(x) = f_0 \varphi_k(x), \quad \varphi_k(x) = \sum_{n=0}^k a_n x^n, \quad (1)$$

where $\varphi_k(x)$ is a dimensionless function with $a_0 = 1$. Writing down the sought function as an asymptotic series

$$f(x) \simeq f_0 \sum_{n=0}^k a_n x^n + \dots, \quad (2)$$

where $x \rightarrow 0$, gives little consolation, since in real problems x is rarely asymptotically small, but usually it is finite and may be even very large. How could we reconstruct the function $f(x)$ for finite values of x from the only knowledge of its asymptotic expansion?

An answer to this question can be provided by the self-similar approximation theory [17–22], with control parameters introduced by means of the fractal transform [23–27,30], defined as

$$F_k(x, s) \equiv x^s f_k(x). \quad (3)$$

This leads to the self-similar exponential and self-similar root approximants. Now, we shall follow a slightly different procedure, which is actually motivated by the very idea of group self-similarity underlying the construction of self-similar approximants.

For a more efficient use of the group self-similarity, we propose to present an initially given asymptotic expansion in the most symmetric way. To this end, we introduce the *factor functions*

$$\varphi_{kp}(x) \equiv 1 + b_{kp} x \quad (p = 1, 2, \dots, k \geq 1). \quad (4)$$

Let us consider the finite series (1) as a polynomial over the field of complex numbers. Then, by the fundamental theorem of algebra [31], such a polynomial can be split in a unique way into a product of the irreducible factors (4), so that

$$\varphi_k(x) = \prod_{p=1}^k \varphi_{kp}(x) \quad (k = 1, 2, \dots). \quad (5)$$

The representation (5) possesses the scaling property, for which if $\varphi_{kp} \rightarrow \lambda \varphi_{kp}$, then $\varphi_k \rightarrow \lambda^k \varphi_k$. Such a scaling property is the simplest rudimental example of functional self-similarity. In this way, the sum (2) can be rewritten as a product

$$f(x) \simeq f_0 \prod_{p=1}^k [\varphi_{kp}(x) + \dots] . \quad (6)$$

Now, instead of accomplishing the self-similar summation for the whole right-hand side of Eq. (2), we may perform it for each factor in the product (6). Thus, we define the fractal transform

$$\Phi_{kp}(x, s) \equiv x^s \varphi_{kp}(x) , \quad (7)$$

construct an approximate cascade whose trajectory is bijective to the sequence $\{\Phi_{kp}(x, s)\}$, embed the cascade into an approximation flow, integrate the flow evolution equation, and realize the inverse fractal transform. All this machinery, with all details, has been expounded in previous papers [17–27], and we therefore do not repeat it here. As a result, each factor (4) can be shown to be renormalized into

$$\varphi_{kp}^*(x) = (1 + A_{kp} x)^{n_{kp}} , \quad (8)$$

where A_{kp} and n_{kp} are control parameters, or simply controls. And a k -order approximation for the sought function $f(x)$ is given by the *self-similar factor approximant*

$$f_k^*(x) = f_0 \prod_{p=1}^k \varphi_{kp}^*(x) . \quad (9)$$

The controls A_{kp} and n_{kp} are determined by expanding the approximant (9) in powers of x and comparing this expansion with the series (2). For short, this can be called a re-expansion procedure, which sometimes is also named the accuracy-through-order relationship. The equations defining the amplitudes A_{kp} and exponents n_{kp} can be cast in the form

$$\sum_{p=1}^k n_{kp} A_{kp}^n = (-1)^{n+1} n b_n , \quad (10)$$

where

$$b_n \equiv \frac{1}{n!} \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \ln \left(\sum_{m=0}^{\infty} a_m x^m \right) . \quad (11)$$

As is easy to check, Eqs. (10) and (11) follow from equating the asymptotic expansions for the logarithms of the factor approximant (9) and of series (2). For each given k , there are $2k$ unknowns in the left-hand side of Eq. (10). Hence, in the series (2), one should have $2k$ nontrivial terms, which makes $n = 1, 2, \dots, 2k$. The series of odd orders $2k + 1$ can also be processed, for which one needs to consider $f(x) - f_0$, instead of $f(x)$. The factor approximants, based on even and odd numbers of terms of a series with alternating signs, often bracket the sought function from below and above. This bracketing is analogous to that occurring for self-similar exponential approximants based on even or odd numbers of asymptotic terms [26,32].

The control parameters may be complex-valued, since, for obtaining the factorized form (5), the sum (1) was treated as a polynomial over the field of complex numbers. But, since the considered function is real, all complex-valued factors should arise in complex conjugate pairs, so that their product be always real.

2.2 Exactly reproducible class of functions

The structure of the self-similar factor approximants (9) suggests that there exists a whole class of functions that are exactly reproducible by means of these approximants. This class is defined as follows. Let $P_n(x)$ be an irreducible polynomial in a real variable x of degree n over the field of real numbers and let α_i and β_j be complex numbers. Compose the real-valued products of powers of such irreducible polynomials as $\prod_i P_{M_i}^{\alpha_i}(x)$ and $\prod_j Q_{N_j}^{\beta_j}(x)$, where $\sum_i M_i = M$ and $\sum_j N_j = N$. This implies that complex powers, if any, always come in complex conjugate pairs. Let these products have no common divisors, such that the ratio

$$f_{MN}^{\alpha\beta}(x) \equiv \frac{\prod_i P_{M_i}^{\alpha_i}(x)}{\prod_j Q_{N_j}^{\beta_j}(x)} \quad (12)$$

be irreducible. Denote by \mathcal{R} a class of functions, which is composed of all products of the forms (12) that play the role of the prime representatives for this class.

Theorem. A function $f(x)$ can be exactly reproduced by the self-similar factor approximants (9) if and only if this function belongs to the class \mathcal{R} , with the prime representatives (12) being exactly reproducible by $f_k^*(x)$ in all orders $k \geq M + N$.

Proof. To prove the proposition for the whole class \mathcal{R} , it is necessary and sufficient to prove it for the prime representatives (12). According to the fundamental theorem of algebra [31], each polynomial in one real variable, over the field of real numbers, can be split into factors of the first and second degree, over the field of real numbers, and into factors of the first degree, over the field of complex numbers. This allows us to split each polynomial entering the prime representative (12) into the product of first-degree factors with complex coefficients. Since, by definition, the ratio (12) is irreducible, it can be presented as a product of $M + N$ first-degree factors. After this, it acquires the form identical to the factor approximant (9) of the order $k = M + N$, taking into account that the exponents n_{kp} in (8) can be negative. Hence the latter pertains to the class \mathcal{R} . And if a function is exactly reproducible by an approximant (9), this function must be from the class \mathcal{R} .

By construction, all parameters (coefficients and powers) of the approximant (9) are defined by equating its asymptotic, in $x \rightarrow 0$, expansion with that of the function (12). An asymptotic expansion in the sense of Poincaré is uniquely defined by the function itself. Two functions, having an identical dependence on the variable and coinciding asymptotic expansions, coincide. That is, a function from the class \mathcal{R} is exactly reproducible by a factor approximant (9). Finally, if a function $f(x)$ exactly equals an approximant $f_k^*(x)$ of the order $k = p$, then the higher approximants, with $k > p$, derived from the same asymptotic expansion of the same function $f(x)$, will coincide with each other.

Remarks. The class \mathcal{R} of the functions, exactly reproducible by means of the self-similar factor approximants, is significantly wider than the class of rational functions that can be

exactly reproduced by Padé approximants. In addition to rational functions, the class \mathcal{R} also includes transcendental functions. Because of this, the self-similar factor approximants should provide a better accuracy for a wider class of functions, as compared to Padé approximants. In what follows, we shall illustrate this by a variety of examples.

For a given real-valued asymptotic series (2) of a real function $f(x)$, the factor approximants (9), by construction, are real in the asymptotic region of $x \rightarrow 0$. However, they may become complex for finite x . If an approximant $f_k^*(x)$ becomes complex for $x > x_k$, this implies that the region of validity of $f_k^*(x)$ is restricted by the interval $[0, x_k]$.

2.3 Relation to the Park method

Factors (8) are appropriate for describing the behavior of functions in the vicinity of critical points. A value $x = x_c$ is termed a critical point of a function $f(x)$ if at this point the latter is either zero, $f(x_c) = 0$, or possesses an algebraic singularity, that is $f(x) \sim (1 - x/x_c)^{-\beta}$ as $x \rightarrow x_c$, where β is positive. This fact was, actually, employed by Park [33] who suggested a method for locating the critical points and calculating the critical exponents. His method is formulated as follows. Assume that: (i) a real function $f(x)$ has a critical point x_c ; (ii) in the neighbourhood of the critical point, the function can be represented as

$$f(x) \simeq f_0 \prod_p (1 - B_p x)^{-\beta_p} \quad (x \rightarrow x_c) ,$$

with all B_p and β_p being real; (iii) the physical critical point corresponds to that which is the closest to the origin, such that, arranging B_p in the descending order of their absolute values, $|B_p| > |B_{p+1}|$, one has $x_c = B_1^{-1}$. Then, defining the coefficients b_n by the expansion

$$\ln f(x) = \sum_{n=1}^{\infty} b_n x^n ,$$

one obtains

$$B_1 = \lim_{n \rightarrow \infty} \frac{b_n}{b_{n-1}} , \quad \beta_1 = \lim_{n \rightarrow \infty} n \frac{b_{n-1}^n}{b_n^{n-1}} .$$

The proof of this statement is based on the generalized Pólya theorem [34], which extends the theorem by Pólya [35], initially proved for entire functions of genus zero, when all $\beta_p = -1$, to the case of real β_p . The Park method for defining the critical points and critical indices is closely related to the Padé analysis of logarithmic derivatives of a series, though in the latter case there is no prior knowledge for the convergence of estimates from Padé approximants [36], which would be analogous to the generalized Pólya theorem.

The principal difference of our approach from the Park method is in the following. First of all, we never require that the sought function be exactly factorizable, but we derive the form (9) as an approximation to this function. Second, we do not impose a constraint that the function must necessarily possess a critical point, and if so, we consider the function not solely in the neighbourhood of the latter, but in the whole region $[0, x_c]$. Third, since we deal with a much more general case, the amplitudes A_{kp} and exponentials n_{kp} are not compulsory real, but may be complex valued.

3 Examples

Any function from the class \mathcal{R} can be reproduced, according to Theorem, exactly, provided that there are enough terms in series (2). However, a reproducible function may be not exactly reproduced when the asymptotic expansion (2) contains not enough terms. In other words, a function to be recovered may be exactly reproducible in principle, but in practice, we may have access to only a few terms in the asymptotic expansion. Then an important question to ask is how well the factor approximants are able to approximate such a function and also it is interesting to observe how the factor approximants converge to the exact result. This problem will be considered in the next subsection.

Those functions that are not from the class \mathcal{R} cannot be reproduced exactly, but they can be very well approximated by the self-similar factor approximants (9), as we illustrate in the following subsections.

3.1 Convergence to exact result

Consider the function

$$f(x) = (1 + 2x)^{3/2}(1 + x)^{1/2}(1 + 0.5x)^{1/3}(1 + 0.1x)^{1/4}, \quad (13)$$

which is from the class \mathcal{R} . Its expansion of eighth order has the coefficients $f_0 = 1$ and

$$\begin{aligned} a_1 &= 3.692, & a_2 &= 3.521, & a_3 &= 0.410, & a_4 &= 0.025, \\ a_5 &= -0.091, & a_6 &= 0.145, & a_7 &= -0.220, & a_8 &= 0.335, \end{aligned}$$

whose behaviour is rather irregular. If we take into account only four terms of series (2), then we get the approximant $f_2^*(x)$, with

$$A_{21} = 1.986 \quad A_{22} = 0.721, \quad n_{21} = 1.562, \quad n_{22} = 0.818.$$

And for the factor approximant $f_3^*(x)$, constructed by means of series (2) of sixth order, we find

$$\begin{aligned} A_{31} &= 2, & A_{32} &= 0.960, & A_{33} &= 0.321, \\ n_{31} &= 1.503, & n_{32} &= 0.583, & n_{33} &= 0.398. \end{aligned}$$

The best Padé approximant that can be built of the sixth-order series (2) is $P_{[4/2]}(x)$, whose accuracy is compared with that of $f_3^*(x)$ in Fig. 1. As is evident, the factor approximant is essentially more accurate, although is not yet exact. But in the next order, we obtain $f_4^*(x)$, with

$$\begin{aligned} A_{41} &= 2, & A_{42} &= 1, & A_{43} &= 0.5, & A_{44} &= 0.1, \\ n_{41} &= \frac{3}{2}, & n_{42} &= \frac{1}{2}, & n_{43} &= \frac{1}{3}, & n_{44} &= \frac{1}{4}, \end{aligned}$$

which coincides with the exact function (13).

3.2 Combination of functions from \mathcal{R} with exponentials

The combination of functions from \mathcal{R} and exponentials are approximated with a very good accuracy. As an example, let us consider

$$f(x) = \left(\frac{1 + Ax + Bx^2}{1 + Cx + Dx^2} \right)^m \exp(-x), \quad (14)$$

The choice of the coefficients and the power m is not important, since the factor multiplying the exponential pertains to the class \mathcal{R} of exactly reproducible functions. For concreteness, let us take $A = 0.5$, $B = 0$, $C = 0.5$, $D = 0.1$, and $m = 0.5$. Then, in the asymptotic series (2), we have $f_0 = 1$ and

$$\begin{aligned} a_1 &= -1.750, & a_2 &= 2.419, & a_3 &= -3.659, \\ a_4 &= 6.060, & a_5 &= -10.499, & a_6 &= 18.622. \end{aligned}$$

The first-order approximant (9) is, clearly, too simple for providing a good approximation for complicated functions. Therefore, here and in what follows, we start the analysis with the second-order approximant. For the considered case, we get in the second order

$$A_{21} = 1.976, \quad A_{22} = -0.077, \quad n_{21} = -0.471, \quad n_{22} = 10.651$$

and in the third order

$$\begin{aligned} A_{31} &= 1.945, & A_{32} &= 0.501, & A_{33} &= 0.000986, \\ n_{31} &= -0.500, & n_{32} &= 0.497, & n_{33} &= -1039. \end{aligned}$$

Both approximants $f_2^*(x)$ and $f_3^*(x)$ perfectly reproduce function (14).

For comparison, we construct the Padé approximants based on the same number of terms in the series (2). Among all possible Padé approximants $P_{[M/N]}(x)$, we select the most accurate for the case considered. Note that the best Padé approximants are not necessarily diagonal. Here, these are $P_{[1/5]}(x)$ and $P_{[2/4]}(x)$. The percentage errors of these approximants, together with the error of $f_3^*(x)$ are shown in Fig. 2. One can observe that the accuracy of $f_3^*(x)$ is incomparably higher than that of the best Padé approximants, whose errors grow fast with x , reaching amplitudes of the order of 100%. In addition, the approximant $P_{[2/4]}(x)$ becomes negative for $x > 5$, which is qualitatively wrong for the positive function (14).

3.3 Exponential multiplied by functions not from the class \mathcal{R}

As an example of a function having no factors from the class \mathcal{R} , let us consider

$$f(x) = \tanh(x) \exp(-x). \quad (15)$$

In its asymptotic series (2), we have $f_0 = 1$ and

$$a_1 = -1, \quad a_2 = \frac{1}{6}, \quad a_3 = \frac{1}{6},$$

$$a_4 = \frac{1}{120}, \quad a_5 = -\frac{31}{360}, \quad a_6 = \frac{1}{5040}.$$

For the second-order approximant (9), we find

$$A_{21} = -0.350 + 0.587 i, \quad A_{22} = A_{21}^*, \quad n_{21} = 0.036 + 0.831 i, \quad n_{22} = n_{21}^*.$$

And for the third order, we obtain

$$A_{31} = -0.00337 + 0.650 i, \quad A_{32} = A_{31}^*, \quad A_{33} = 0.071,$$

$$n_{31} = -0.871 + 0.023 i, \quad n_{32} = n_{31}^*, \quad n_{33} = -13.777.$$

Again, we compare $f_3^*(x)$ with the most accurate Padé approximants that for this case are $P_{[1/5]}(x)$ and $P_{[2/4]}(x)$. The corresponding percentage errors are shown in Fig. 3. Again, we see that $f_3^*(x)$ has much higher accuracy than the best Padé approximants. The most accurate of the latter, $P_{[1/5]}(x)$, becomes negative for $x > 6$, which is qualitatively wrong.

A combination of a logarithm and an exponential yields a non-monotone function

$$f(x) = \ln(1+x) \exp(-x). \quad (16)$$

The coefficients of the related asymptotic expansion (2) are $f_0 = 1$ and

$$a_1 = -\frac{3}{2}, \quad a_2 = \frac{4}{3}, \quad a_3 = -1,$$

$$a_4 = \frac{89}{120}, \quad a_5 = -\frac{83}{144}, \quad a_6 = \frac{593}{1260}.$$

For the parameters of the factor approximant $f_2^*(x)$, we get

$$A_{21} = 0.930, \quad A_{22} = 0.013, \quad n_{21} = -0.466, \quad n_{22} = -83.334,$$

and for those of $f_3^*(x)$, we find

$$A_{31} = 0.973, \quad A_{32} = 0.574, \quad A_{33} = 0.00212,$$

$$n_{31} = -0.363, \quad n_{32} = -0.215, \quad n_{33} = -483.556.$$

The most accurate Padé approximant here is $P_{[1/5]}(x)$. The corresponding percentage errors are presented in Fig. 4, from where it is seen that $f_3^*(x)$ is significantly more accurate than $P_{[1/5]}(x)$.

3.4 Functions not from the class \mathcal{R} which converge to a constant at infinity

The previous functions converge to zero at infinity. Let us now consider the function

$$f(x) = \exp\left(1 - \frac{1}{\sqrt{1+x}}\right), \quad (17)$$

which increases at infinity to a finite value. In the expansion (2), we have $f_0 = 1$ and

$$\begin{aligned} a_1 &= 0.5 , & a_2 &= -0.25 , & a_3 &= 0.146 , & a_4 &= -0.091 , \\ a_5 &= 0.059 , & a_6 &= -0.038 , & a_7 &= 0.025 , & a_8 &= -0.016 . \end{aligned}$$

For the second-order factor approximant, we find

$$A_{21} = 0.570 , \quad A_{22} = 1.097 , \quad n_{21} = -0.671 , \quad n_{22} = 0.805 ,$$

and for $f_3^*(x)$, we have

$$\begin{aligned} A_{31} &= 1.041 , & A_{32} &= 0.794 , & A_{33} &= 0.265 , \\ n_{31} &= 1.243 , & n_{32} &= -0.922 , & n_{33} &= -0.235 . \end{aligned}$$

The best Padé approximant $P_{[2/2]}(x)$ is less accurate than $f_3^*(x)$, as is shown in Fig. 5. The accuracy becomes even better for $f_4^*(x)$, for which

$$\begin{aligned} A_{41} &= 0.151 , & A_{42} &= 1.023 , & A_{43} &= 0.881 , & A_{44} &= 0.516 , \\ n_{41} &= -0.150 , & n_{42} &= 1.675 , & n_{43} &= -1.195 , & n_{44} &= -0.267 . \end{aligned}$$

Notice that, passing to higher approximations, the sum $\sum_p n_{kp}$ decreases,

$$\begin{aligned} n_{21} + n_{22} &= 0.133 , & n_{31} + n_{32} + n_{33} &= 0.086 , \\ n_{41} + n_{42} + n_{43} + n_{44} &= 0.063 , \end{aligned}$$

which is the correct trend, since in the limit it should be

$$\lim_{k \rightarrow \infty} \sum_{p=1}^k n_{kp} = 0 .$$

The accuracy of the factor approximants can also be improved by assuming the validity of the condition $\sum_p n_{kp} = 0$ for finite k .

3.5 Classical example of ϕ^4 theory with strongly divergent series

A classical example of a strongly divergent series is provided by the asymptotic expansion, in powers of a coupling parameter, of generating functionals in field theory or of partition functions in statistical mechanics. The generic structure of such divergent expansions is exemplified by expanding, in powers of the coupling g , the generating functional

$$I(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\varphi^2 - g\varphi^4) d\varphi , \quad (18)$$

typical of φ^4 field theory. In the asymptotic series (2), with g instead of x , the coefficients are $f_0 = 1$ and

$$a_n = \frac{(-1)^n}{\sqrt{\pi} n!} \Gamma\left(2n + \frac{1}{2}\right) .$$

The latter grows with $n \rightarrow \infty$ as $a_n \sim n^n$. Such a series is divergent for any finite g . For the second-order approximant $I_2^*(g)$, we find

$$A_{21} = 19.141, \quad A_{22} = 4.859, \quad n_{21} = -0.00862, \quad n_{22} = -0.120.$$

And for $I_3^*(g)$, we obtain

$$\begin{aligned} A_{31} &= 31.220, & A_{32} &= 13.317, & A_{33} &= 3.464, \\ n_{31} &= -0.000526, & n_{32} &= -0.022, & n_{33} &= -0.125. \end{aligned}$$

The most accurate Padé approximant, constructed with the same number of the asymptotic terms, is $P_{[3/3]}(g)$. This is compared with $I_3^*(g)$ in Fig. 6.

Again, we see that the accuracy of the factor approximant highly surpasses that of the Padé approximant. This is especially noticeable for large couplings g , where the Padé approximant completely fails. Really, $P_{[3/3]}(g)$ is finite for $g \rightarrow \infty$, which is in contradiction with the behavior of $I(g)$ at large g , where the integral (17) tends to zero,

$$I(g) \simeq 1.023 g^{-0.25} \quad (g \rightarrow \infty). \quad (19)$$

The factor approximants also decrease in the strong-coupling limit as

$$I_2^*(g) \simeq 0.806 g^{-0.129}, \quad I_3^*(g) \simeq 0.807 g^{-0.148} \quad (g \rightarrow \infty). \quad (20)$$

Let us emphasize that all parameters of the self-similar factor approximants are defined only through the weak-coupling expansion. And it looks almost mysterious that they can reasonably extrapolate the behavior at the strong-coupling limit. The accuracy of the approximants (9) increases with their order. Thus, for the approximant $I_4^*(g)$, we arrive at the parameters

$$\begin{aligned} A_{41} &= 43.965, & A_{42} &= 23.064, & A_{43} &= 10.294, & A_{44} &= 2.677, \\ n_{41} &= -0.000025, & n_{42} &= -0.00259, & n_{43} &= -0.035, & n_{44} &= -0.124. \end{aligned}$$

In the strong-coupling limit, this yields

$$I_4^*(g) \simeq 0.810 g^{-0.161} \quad (g \rightarrow \infty). \quad (21)$$

Note that the diagonal Padé approximants are always finite at infinity, thus allowing the approximation of only a very narrow class of functions [37]. Contrary to this, the factor approximants immediately catch the correct behavior at infinity; their accuracy increasing with the approximation order. Hence, the self-similar factor approximants, being based on an asymptotic expansion at zero, can correctly reproduce the behavior of the sought function for the whole range of its variable, including the behavior at infinity.

3.6 Expansion factor of a three-dimensional polymer chain

As another physical example, let us consider the expansion factor $\alpha(z)$ for a three-dimensional polymer chain with excluded-volume interaction, where z is a dimensionless coupling parameter [38,39]. An asymptotic series of the form (2), derived by means of perturbation theory [38], yields the coefficients $f_0 = 1$ and

$$a_1 = \frac{4}{3} , \quad a_2 = -2.075385396 , \quad a_3 = 6.296879676 ,$$

$$a_4 = -25.05725072 , \quad a_5 = 116.134785 , \quad a_6 = -594.71663 .$$

In the strong-coupling limit, it has been established numerically [37,38] that $\alpha(z)$ can be accurately represented by

$$\alpha(z) \simeq 1.531 z^{0.3544} + 0.184 z^{-0.5756} \quad (z \rightarrow \infty) . \quad (22)$$

For the approximant $\alpha_2^*(z)$, we have

$$A_{21} = 6.064 , \quad A_{22} = 2.962 , \quad n_{21} = 0.105 , \quad n_{22} = 0.235 ,$$

with the strong-coupling limit

$$\alpha_2^*(z) \simeq 1.560 z^{0.340} + 0.151 z^{-0.660} \quad (z \rightarrow \infty) \quad (23)$$

being very close to the exact numerical value. In the next order, the parameters of $\alpha_3^*(z)$ are

$$\begin{aligned} A_{31} &= 7.019 , & A_{32} &= 4.635 , & A_{33} &= 2.262 , \\ n_{31} &= 0.033 , & n_{32} &= 0.164 , & n_{33} &= 0.151 . \end{aligned}$$

And the strong-coupling limit gives

$$\alpha_3^*(z) \simeq 1.551 z^{0.348} + 0.166 z^{-0.652} \quad (z \rightarrow \infty) . \quad (24)$$

Both the coefficients as well as the powers of the strong-coupling divergence are very accurate, as compared to the numerically found behavior of $\alpha(z)$, with the percentage error of 1%. The best Padé approximant here is $P_{[3/3]}(z)$. In Fig. 7, the percentage errors of $\alpha_3^*(z)$ and $P_{[3/3]}(z)$ are shown, compared with the numerically fitted [39] equation

$$\alpha(z) = \left(1 + 7.524 z + 11.06 z^2 \right)^{0.1172} . \quad (25)$$

As is evident, the factor approximant dramatically outperforms the Padé approximant. The latter completely fails at large $z \rightarrow \infty$, where it is finite, while $\alpha(z)$ diverges. To the contrary, the factor approximant $\alpha_3^*(z)$ possesses the correct behavior in the strong-coupling limit. Moreover, not only the main asymptotic, as $z \rightarrow \infty$, term is very accurate, but the next term, describing the so-called correction to scaling, is also of good accuracy. Of special interest is the strong-coupling exponent

$$\nu \equiv \frac{1}{2} + \frac{1}{4} \lim_{z \rightarrow \infty} \frac{\ln \alpha(z)}{\ln z} , \quad (26)$$

which is a kind of a critical index for the polymer chain [40]. According to the Muthukumar and Nickel [38,39] numerical estimate, $\nu_{MN} = 0.5886$. For the self-similar factor approximants, we have

$$\nu_2^* = 0.585 , \quad \nu_3^* = 0.587 .$$

Recent numerical estimates [40,41] give the value $\nu = 0.5877 \pm 0.0006$, which is very close to ν_3^* .

3.7 Nonlinear differential equation for state-dependent solid friction

Let us now demonstrate how the self-similar factor approximants can be employed for solving nonlinear differential equations. Let us consider the Ruina-Dieterich law of solid friction between two solid surfaces sliding against each other (see [42,43]). The Ruina-Dieterich law involves the so-called state variable denoted here as f in a dimensionless form, which is usually thought to quantify the true area of contacts of the asperities of two solid surfaces. The state variable f obeys the following simple non-linear differential equation (put in dimensionless form both for f and t)

$$\frac{df}{dt} = \beta - f^{1-m} , \quad (27)$$

where β and m are parameters. Equation (27) possesses two qualitatively different types of solutions corresponding to $m < 1$ and $m > 1$.

Consider, first, the case $m < 1$. Let $m = 0.85$, $\beta = 0.526$, and the initial condition $f(0) = 0.5$. Equation (27) allows us to derive the short-time expansion for $f(t)$, presented as the series (2) in powers of time t , simply by using the Taylor expansion formula and by taking successive derivatives of (27). This gives $f_0 = f(0)$ and

$$a_1 = -0.750 , \quad a_2 = 0.102 , \quad a_3 = 0.012 ,$$

$$a_4 = 0.00226 , \quad a_5 = 0.000455 , \quad a_6 = 0.0000861 .$$

By the standard procedure, we obtain the factor approximant $f_2^*(t)$, with

$$A_{21} = -0.391 + 0.116 i , \quad A_{22} = A_{21}^* , \quad n_{21} = 0.682 + 0.932 i , \quad n_{22} = n_{21}^* .$$

And the parameters for $f_3^*(t)$ are

$$A_{31} = -0.430 + 0.118 i , \quad A_{32} = A_{31}^* , \quad A_{33} = -0.149 ,$$

$$n_{31} = 0.790 + 0.471 i , \quad n_{32} = n_{31}^* , \quad n_{33} = -0.272 .$$

The accuracy of the approximants can be checked against the direct numerical solution of Eq. (27). In Fig. 8, we compare the percentage errors of $f_3^*(t)$ with those of the best Padé approximants $P_{[2/2]}(t)$ and $P_{[3/3]}(t)$. As we see, the factor approximant is considerably more accurate.

4 Critical Behavior

Solutions to physical problems often display a critical behavior, when a function $f(x)$ tends, at a critical point x_c , either to zero or to infinity. Self-similar factor approximants make it possible to describe these two types of critical behavior, providing accurate estimates for the critical points as well as for critical indices. If one is interested solely in the critical behavior, then our approach reduces to the Park method, as is discussed in Sec. II. However, we would like to stress that the factor approximants not solely describe well the critical neighbourhood, but provide an accurate approximation in the whole region $[0, x_c]$. It is also important to realize that, in order to achieve an accurate description, one does not necessarily require the knowledge of a large number of terms in an asymptotic series, but only a few terms are often sufficient. This applies to both types of critical points, either zeros or singularities.

4.1 Critical behavior near zero

An example of the first type is the function

$$f(x) = (\cos \sqrt{x})^{1/3}, \quad (28)$$

tending to zero at the critical point $x_c = \pi^2/4 = 2.467$ with the critical index $1/3$. the coefficients of the asymptotic series (2) are $f_0 = 1$ and

$$\begin{aligned} a_1 &= -0.167, & a_2 &= -0.014, & a_3 &= -0.00355, \\ a_4 &= -0.000982, & a_5 &= -0.000295, & a_6 &= -0.0000937. \end{aligned}$$

Notice that all coefficients here are negative. Such asymptotic series with all coefficients of the same sign are known to be very difficult for any kind of resummation procedure [2]. But, with the factor approximants (9), there is no problem in approximating the sought function in the whole region from $x = 0$ to the critical point. For the approximant $f_2^*(x)$, we get

$$A_{21} = -0.405, \quad A_{22} = -0.024, \quad n_{21} = 0.334, \quad n_{22} = 1.333.$$

In this approximation, the critical point $x_c = |A_{21}|^{-1} = 2.469$ is already close to the exact value $\pi^2/4 = 2.467$. The same quality of results is obtained for the critical index n_{21} , which almost coincides with the exact index $1/3$. For the next approximation $f_3^*(x)$, we find

$$\begin{aligned} A_{31} &= -0.405, & A_{32} &= -0.044, & A_{33} &= -0.00541, \\ n_{31} &= 0.333, & n_{32} &= 0.375, & n_{33} &= 2.791. \end{aligned}$$

The approximant $f_3^*(x)$ is practically indistinguishable from function (28) in the whole region $[0, x_c]$. The critical point $x_c = |A_{31}|^{-1} = 2.467$ yields the exact value $\pi^2/4$ with a precision of about 10^{-3} . The corresponding critical index n_{31} differs from the exact index in the seventh decimal digit. The best Padé approximant $P_{[3/2]}(x)$ is much less accurate, giving the critical point $x_c = 2.507$ and a very inaccurate critical index equal to 1 instead of $1/3$. The accuracies of $f_3^*(x)$ and of $P_{[3/2]}(x)$ are compared in Fig. 9.

4.2 Critical singular behavior

In several physical problems [42], the critical behavior is described by the function

$$f(x) = \frac{\pi}{2\arccos x} , \quad (29)$$

which diverges at the critical point $x_c = 1$, with the critical index $1/2$. The asymptotic series (2) has the coefficients $f_0 = 1$ and

$$\begin{aligned} a_1 &= 0.637 , & a_2 &= 0.405 , & a_3 &= 0.364 , \\ a_4 &= 0.299 , & a_5 &= 0.281 , & a_6 &= 0.248 . \end{aligned}$$

This is again the case of a series with constant-sign coefficients, which is a difficult problem for resummation. For the approximant $f_2^*(x)$, we find

$$A_{21} = -0.990 , \quad A_{22} = 0.778 , \quad n_{21} = -0.514 , \quad n_{22} = 0.164 .$$

This gives the critical point $x_c = |A_{21}|^{-1} = 1.010$ and the critical index $|n_{21}|$ in a very good agreement with the exact values. In the next order, with an accuracy up to three decimal digits, we obtain

$$\begin{aligned} A_{31} &= -1 , & A_{32} &= 0.912 , & A_{33} &= 0.363 , \\ n_{31} &= -0.501 , & n_{32} &= 0.091 , & n_{33} &= 0.146 . \end{aligned}$$

The critical point $x_c = |A_{31}|^{-1} = 1$ coincides with the exact value, and the critical index $|n_{31}|$ is also practically equal to the exact index $1/2$. The best Padé approximant, $P_{[4/1]}(x)$, is much less accurate, yielding the critical point $x_c = 1.064$ and a bad estimate for the critical index, equal to 1 instead of $1/2$. The accuracy of the approximants $f_3^*(x)$ and $P_{[4/1]}(x)$ are compared in Fig. 10.

4.3 Critical behavior in nonlinear differential equation

Critical behavior may arise in solutions of differential equations. For instance, the Ruina-Dieterich law of solid friction, given by Eq. (27), is qualitatively different for $m < 1$ and $m > 1$. For $m < 1$, the solution has no zeros. But for $m > 1$, the solution becomes zero at a critical time t_c approached with the critical index $1/m$. Let $m = 1.5$ and all other parameters be the same as for the noncritical case considered in the previous section. Then the critical time is $t_c = 0.33004$. For the short-time expansion in powers of time t , we have $f_0 = f(0) = 0.5$ and

$$\begin{aligned} a_1 &= -1.776 , & a_2 &= -1.256 , & a_3 &= -1.706 , \\ a_4 &= -3.024 , & a_5 &= -6.114 , & a_6 &= -13.388 . \end{aligned}$$

The asymptotic series strongly diverges. Constructing the factor approximant $f_2^*(t)$, we get

$$A_{21} = -3.049 , \quad A_{22} = -0.558 , \quad n_{21} = 0.616 , \quad n_{22} = -0.181 .$$

This gives the critical time $t_c = |A_{21}|^{-1} = 0.328$ and the critical index n_{21} , estimating rather well the corresponding exact values $t_c = 0.330$ and $1/m = 0.667$, respectively. For the parameters of $f_3^*(t)$, we derive

$$A_{31} = -3.036, \quad A_{32} = -1.676, \quad A_{33} = -0.225, \\ n_{31} = 0.630, \quad n_{32} = -0.046, \quad n_{33} = -0.269.$$

The critical time $t_c = |A_{31}|^{-1} = 0.3294$ approximates the exact numerical value with a good accuracy, the error being only -0.19% . The related critical index n_{31} is also close to the exact index 0.667. The most accurate Padé approximant $P_{[3/3]}(t)$ yields a much worse approximation, with the critical time $t_c = 0.34134$ and the critical index 1 instead of $2/3$. The relative accuracies of the approximants $f_3^*(x)$ and $P_{[3/3]}(x)$ are demonstrated in Fig. 11.

4.4 Two examples of critical phenomena in statistical physics

An analysis of critical behavior would not be complete without considering critical phenomena of statistical mechanics. Consider, for instance, the three-dimensional spin-1/2 Ising model with a simple cubic lattice. Thermodynamic characteristics of this model can be presented in the form of high-temperature expansions in powers of the parameter $v \equiv \tanh(J/k_B T)$, where J is the exchange integral and T , the temperature. For example, the second derivative of the susceptibility with respect to an external field

$$\chi_0^{(2)} = \left. \frac{\partial^2 \chi}{\partial H^2} \right|_{H=0} \quad (30)$$

is known [44] as the series expansion containing the powers of v up to v^{17} . The first coefficients of this expansion are $f_0 = -2$ and

$$a_1 = 24, \quad a_2 = 318, \quad a_3 = 3240, \quad a_4 = 28158, \\ a_5 = 220680, \quad a_6 = 1604406, \quad a_7 = 11029560, \quad a_8 = 72559422.$$

The coefficients are of constant sign and are growing fast. Applying the method of the factor approximants, we have in the second order

$$A_{21} = -4.602, \quad A_{22} = 7.373, \quad n_{21} = -4.300, \quad n_{22} = 0.571,$$

where we limit ourselves by three decimal digits. This gives the critical point $v_c = |A_{21}|^{-1} = 0.217$ at which the function diverges with the critical index $|n_{21}|$. These values can be compared with accurate Monte Carlo calculations [45] yielding the critical point $v_c = 0.218092$ and with a refined analysis by means of integral approximants [44] giving the critical index 4.37. In the third order, again with an accuracy up to three decimals, we obtain

$$A_{31} = -4.601, \quad A_{32} = 7.375, \quad A_{33} = -44.934, \\ n_{31} = -4.301, \quad n_{32} = 0.571, \quad n_{33} = 0.$$

Then the critical point is $v_c = |A_{31}|^{-1} = 0.217$ and the critical index is $|n_{31}|$, which are practically the same as in the second approximation. The best Padé approximant $P_{[1/4]}(v)$ results in rather inaccurate values for the critical point $v_c = 0.153$ and the critical index equal to 1. The fourth-order factor approximant does not lead to a noticeable change of the critical parameters, as compared to the third-order one, also giving $v_c = 0.217$ and the critical index 4.301. The percentage errors of those estimates are about 1%.

As another example of high-temperature series expansions, let us consider such series for the $(2+1)$ -dimensional Ising model, defined by the Hamiltonian

$$H = \sum_i (1 - \sigma_i^3) - x \sum_{\langle ij \rangle} \sigma_i^1 \sigma_j^1 - h \sum_i \sigma_i^1 , \quad (31)$$

in which σ_i^α , with $\alpha = 1, 2, 3$, are Pauli matrices; the index i enumerates the sites on the two-dimensional spatial lattice; $\langle ij \rangle$ denotes nearest-neighbour pairs of sites; x is an effective coupling parameter corresponding to the inverse temperature in the Euclidean formulation of field theory; and h stands for magnetic field. Here, we consider the triangular lattice. Let us take, e.g., a series in powers of x for the mass gap F at zero magnetic field,

$$F \equiv E_1 - E_0 = F(x) , \quad (32)$$

which is the difference between the energy of the first excited level and the ground-state energy. The series (2) for this case [46] has the coefficients $f_0 = 2$ and

$$\begin{aligned} a_1 &= -3 , & a_2 &= -3 , & a_3 &= -5.25 , & a_4 &= -15.75 , \\ a_5 &= -49.265625 , & a_6 &= -173.3554688 , & a_7 &= -627.602783 , & a_8 &= -2397.718506 . \end{aligned}$$

For the parameters of the second-order factor approximant $F_2^*(x)$, we find

$$A_{21} = -4.7404 , \quad A_{22} = 1.7404 , \quad n_{21} = 0.6582 , \quad n_{22} = 0.0691 .$$

The critical point, following from here, is $x_c = |A_{21}|^{-1} = 0.2110$. The mass gap tends to zero at x_c as $F \sim (x_c - x)^\nu$, with the critical index $\nu = n_{21}$. For the third-order approximant $F_3^*(x)$, we have

$$\begin{aligned} A_{31} &= -4.7826 , & A_{32} &= -3.5899 , & A_{33} &= 1.5136 , \\ n_{31} &= 0.6211 , & n_{32} &= 0.0458 , & n_{33} &= 0.0890 . \end{aligned}$$

Hence, the critical point is $x_c = |A_{31}|^{-1} = 0.2091$ and the critical index is $\nu = n_{31}$. In the next order, for the parameters of $F_4^*(x)$, we obtain

$$\begin{aligned} A_{41} &= -4.7629 , & A_{42} &= -2.1880 , & A_{43} &= 3.9403 , & A_{44} &= 1.2670 , \\ n_{41} &= 0.6445 , & n_{42} &= 0.0375 , & n_{43} &= 0.0007 , & n_{44} &= 0.1178 . \end{aligned}$$

This results in the critical point $x_c = |A_{41}|^{-1} = 0.2100$ and the critical index $\nu = n_{41}$. These values can be compared with those summarized in Ref. [44], where x_c varies between 0.2097 and 0.2098, while the index ν is between 0.627 and 0.641, which is very close to our results.

5 Conclusion

By employing the self-similar approximation theory, we have derived a new class of approximants, which, because of their form, we call the *self-similar factor approximants*. All control parameters of this class of approximants are defined by means of the accuracy-through-order relationship. These approximants reproduce *exactly* a wide class of functions from the sole knowledge of their asymptotic expansions. For other functions not from the exactly reproducible class, the factor approximants provide a very high accuracy, which is essentially higher than that given by the best Padé approximants constructed with the same number of asymptotic terms.

The self-similar factor approximants are able to reproduce with a good accuracy various kinds of functions, diminishing and increasing, monotone and non-monotone, on finite or infinite intervals. The asymptotic series, used for constructing the factor approximants, can have their coefficients with alternating signs or with constant signs. The series can also be strongly divergent. Although the factor approximants are derived from asymptotic series for a variable in the vicinity of zero, they extrapolate well to the behavior of the sought functions at infinity. The approximants are able to predict the occurrence of critical phenomena, providing accurate values for both the critical points and critical indices.

Note that a natural generalization involves combining the factor approximants with the self-similar root approximants and exponential approximants. The latter can be treated as a limiting case of the factor approximants, since the renormalized factor function (8) tends, as $n_{k_p} \rightarrow \infty$, to an exponential. The self-similar exponential approximants, as has been recently shown [47], enjoy the property of exactly reconstructing exponential functions. Therefore, the factor approximants, together with their limiting exponential forms, and being combined with the self-similar root approximants, provide a powerful tool for an accurate reconstruction of a very wide class of functions.

At first glance, it may appear mysterious that, knowing solely the behavior of a function in the vicinity of zero, it is possible to correctly predict its behavior at infinity or near a critical point. However, there is no miracle here. The coefficients of an asymptotic series contain a great deal of information about their parent function, provided that the latter are differentiable up to sufficiently high order. Then, the main problem is that this information is hidden, encoded. And one needs to possess a guide and a key for decoding that information. The idea of *group self-similarity* for subsequent approximations [17–27] serves as a guide pointing at the general properties of the sought function, which allows for the extrapolation of the given approximations. And the self-similar factor approximants is a practical key for realizing this extrapolation.

Since the self-similar approximation method presented here is capable of capturing the features of rather complex functions with very good accuracy from the knowledge of a few numerical coefficients, one can view this approach as a complexity reduction scheme, or better as an encoding-decoding scheme. In the language of algorithmic complexity theory [48], the differentiable functions with only isolated critical points have a low degree of complexity.

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Figure Captions

Fig. 1. Percentage errors of the self-similar factor approximant $f_3^*(x)$ (solid line) and of the Padé approximant $P_{[4/2]}(x)$ (dotted line), as compared with function (13).

Fig. 2. Percentage errors of the self-similar factor approximant $f_3^*(x)$ (solid line) and of the Padé approximants $P_{[1/5]}(x)$ (dashed line) and $P_{[2/4]}(x)$ (dotted line), compared with function (14).

Fig. 3. Percentage errors of the factor approximant $f_3^*(x)$ (solid line) and of the Padé approximants $P_{[1/5]}(x)$ (dotted line) and $P_{[2/4]}(x)$ (dashed line), compared with function (15).

Fig. 4. Percentage errors of $f_3^*(x)$ (solid line) and $P_{[1/5]}(x)$ (dashed line), approximating function (16).

Fig. 5. Percentage errors of $f_3^*(x)$ (solid line) and $P_{[2/2]}(x)$ (dotted line), compared with function (17).

Fig. 6. Percentage errors of $I_3^*(g)$ (solid line) and $P_{[3/3]}(g)$ (dashed line), compared with the numerical values of integral (18).

Fig. 7. Percentage errors of the approximants for the expansion factor $\alpha_3^*(z)$ (solid line) and $P_{[3/3]}(z)$ (dashed line), compared with numerical values.

Fig. 8. Percentage errors of the approximants for friction, $f_3^*(t)$ (solid line), $P_{[2/2]}(t)$ (dotted line), and $P_{[3/3]}(t)$ (dashed line), compared with an exact numerical solution of Eq. (27).

Fig. 9. Percentage errors of $f_3^*(x)$ (solid line) and $P_{[3/2]}(x)$ (dotted line) as compared to function (28).

Fig. 10. Percentage errors of $f_3^*(x)$ (solid line) and $P_{[4/1]}(x)$ (dotted line) with respect to function (29).

Fig. 11. Percentage errors of $f_3^*(x)$ (solid line) and $P_{[3/3]}(x)$ (dotted line), compared with the numerical solution of Eq. (27).

Fig. 1

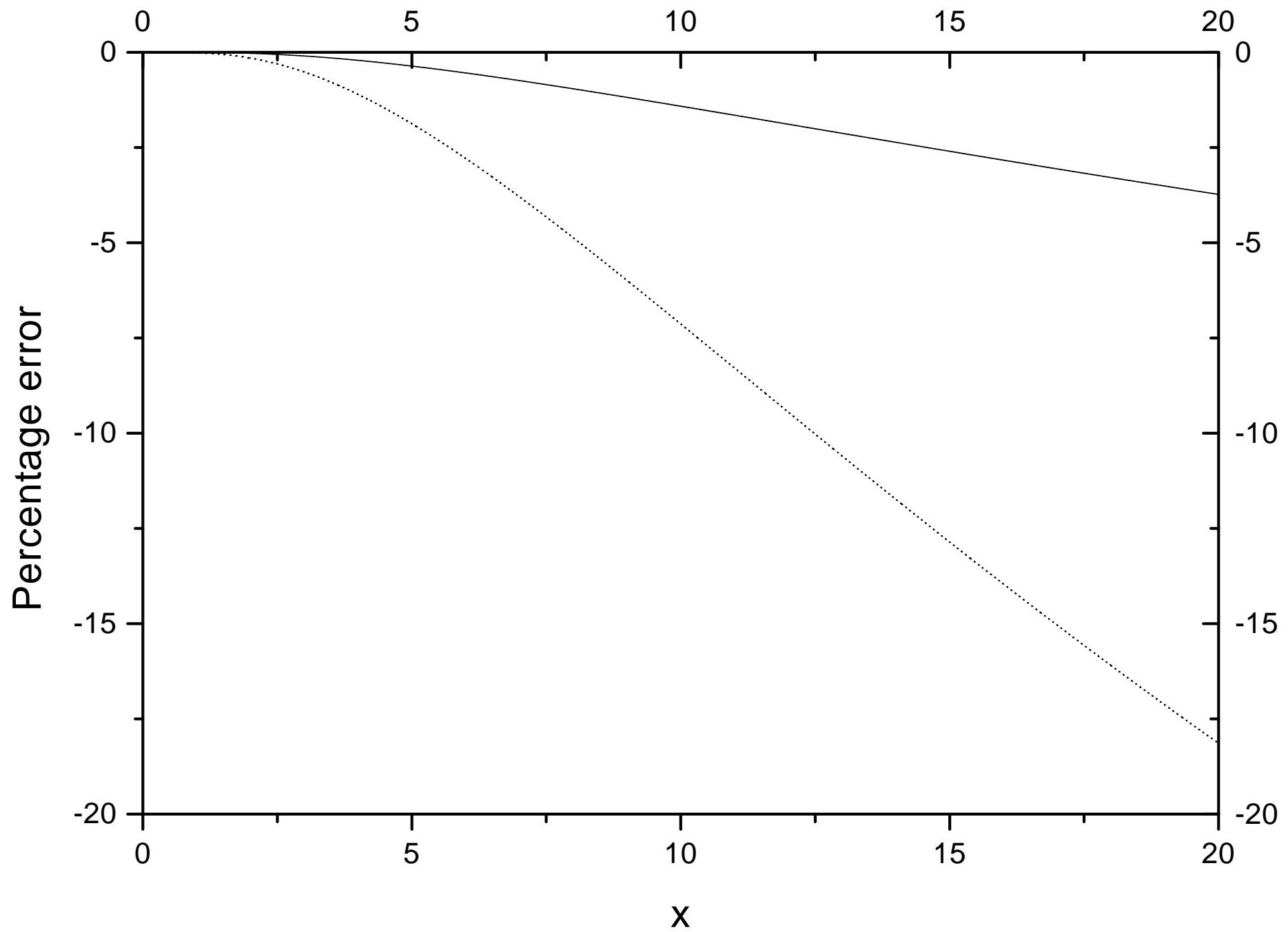


Fig. 2

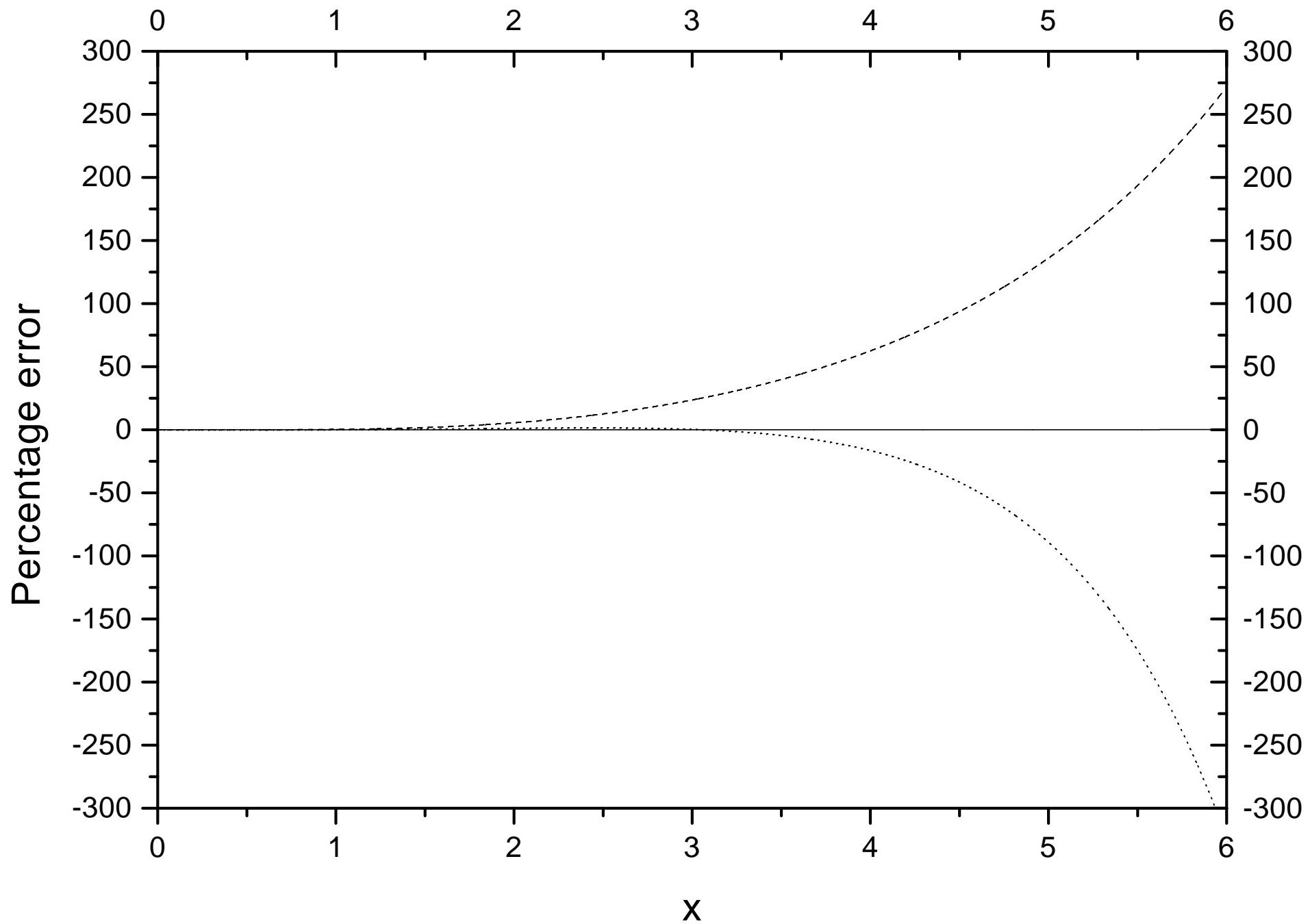


Fig. 3

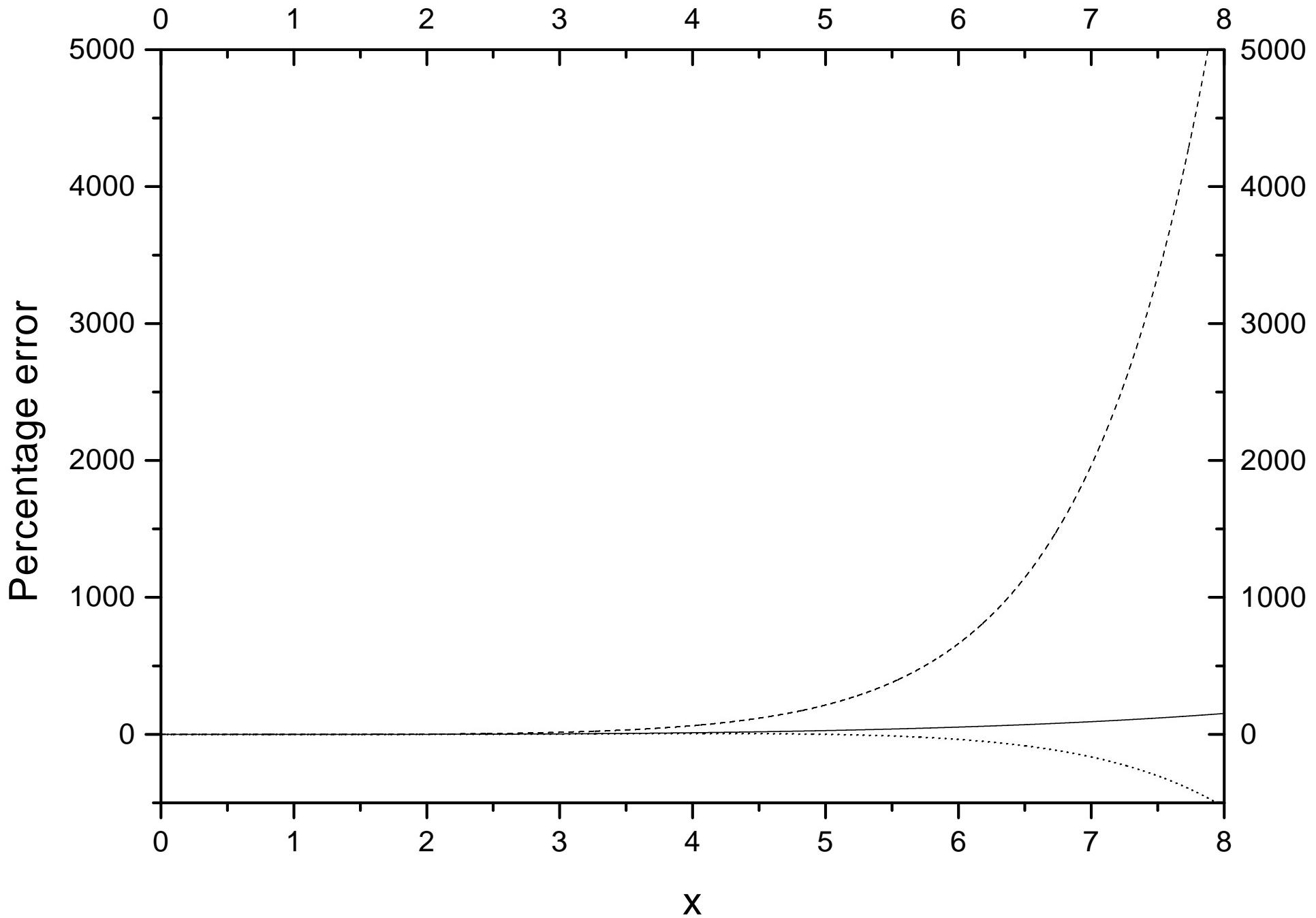


Fig. 4

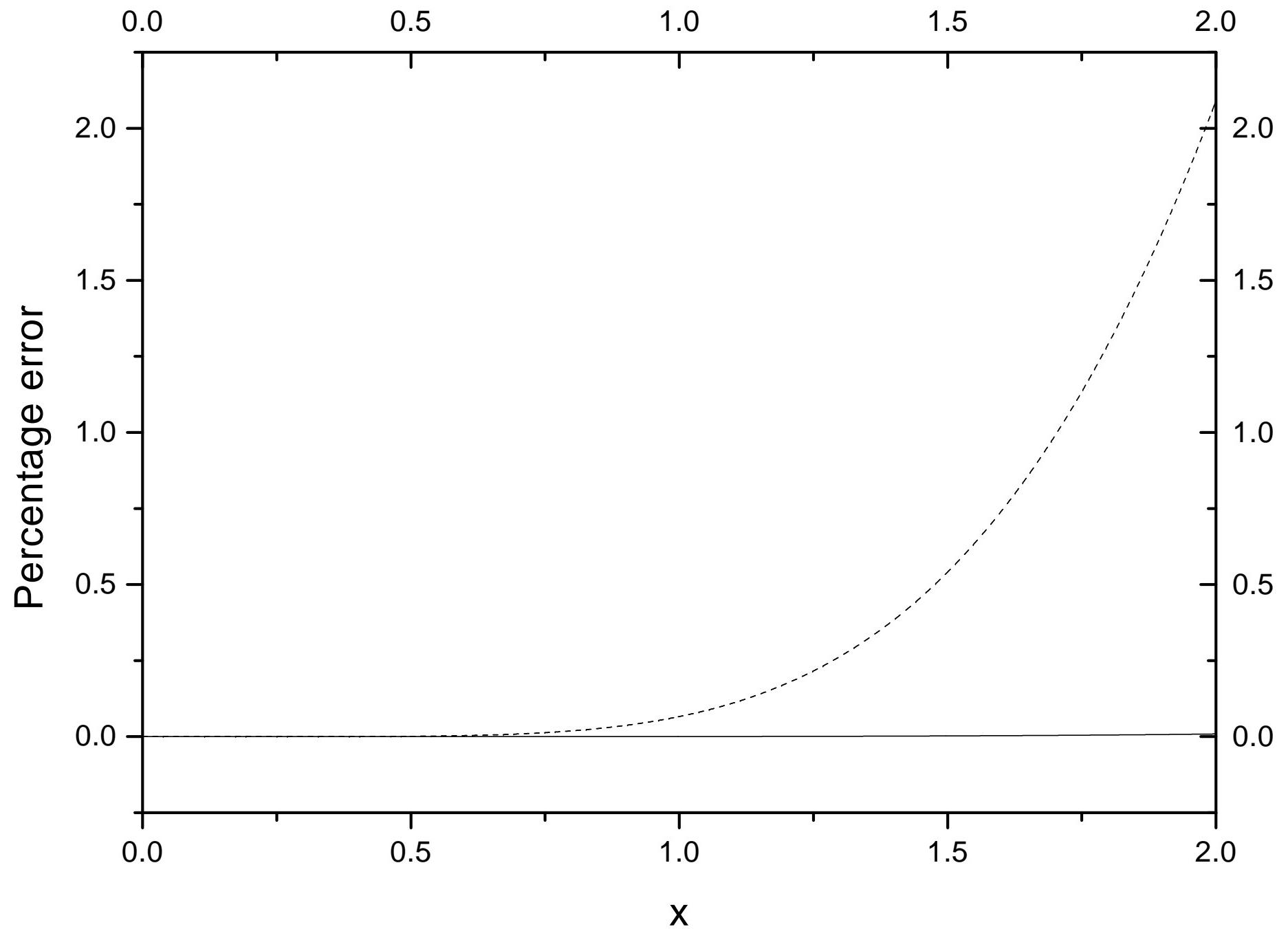


Fig. 5

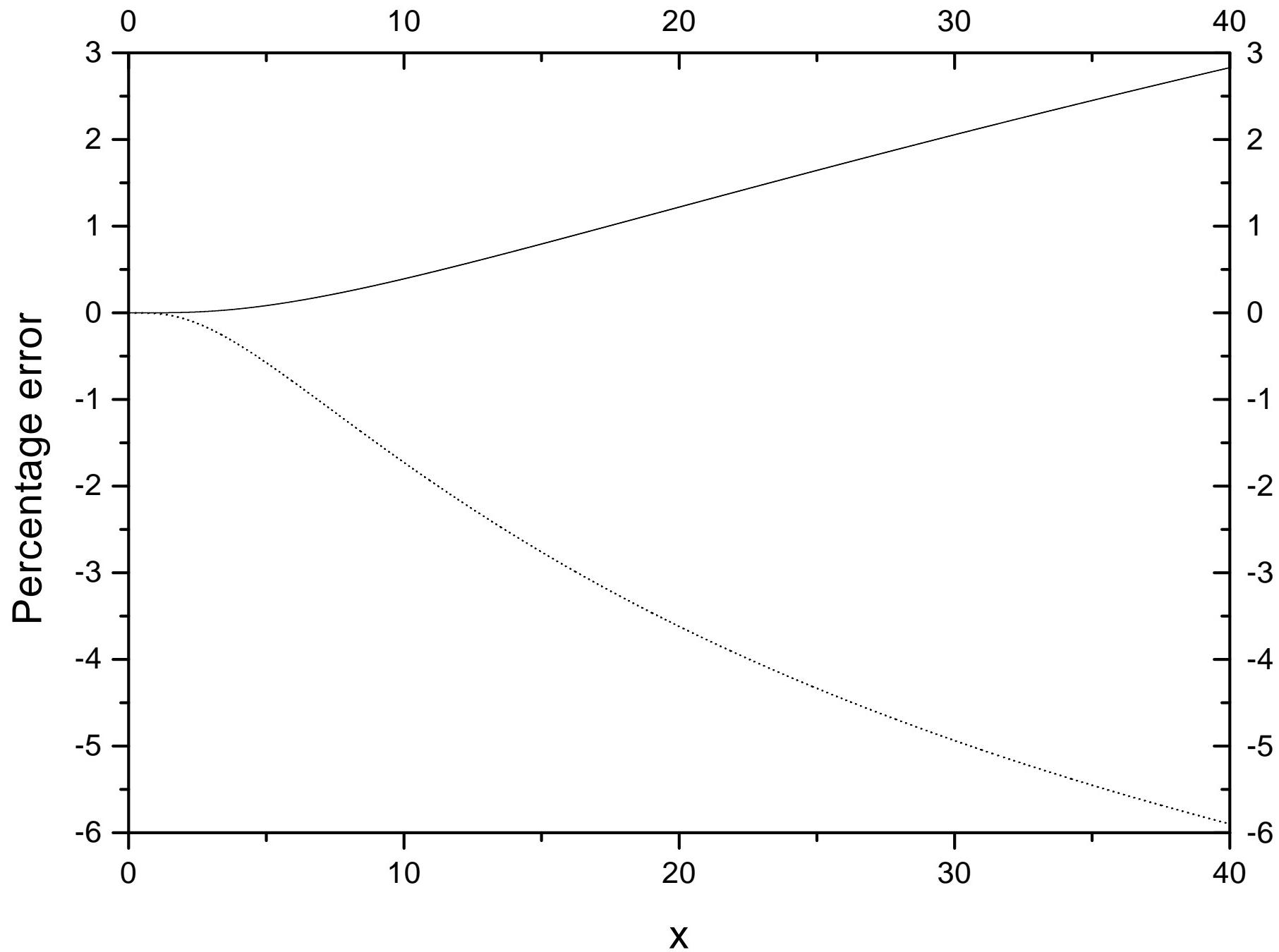


Fig. 6

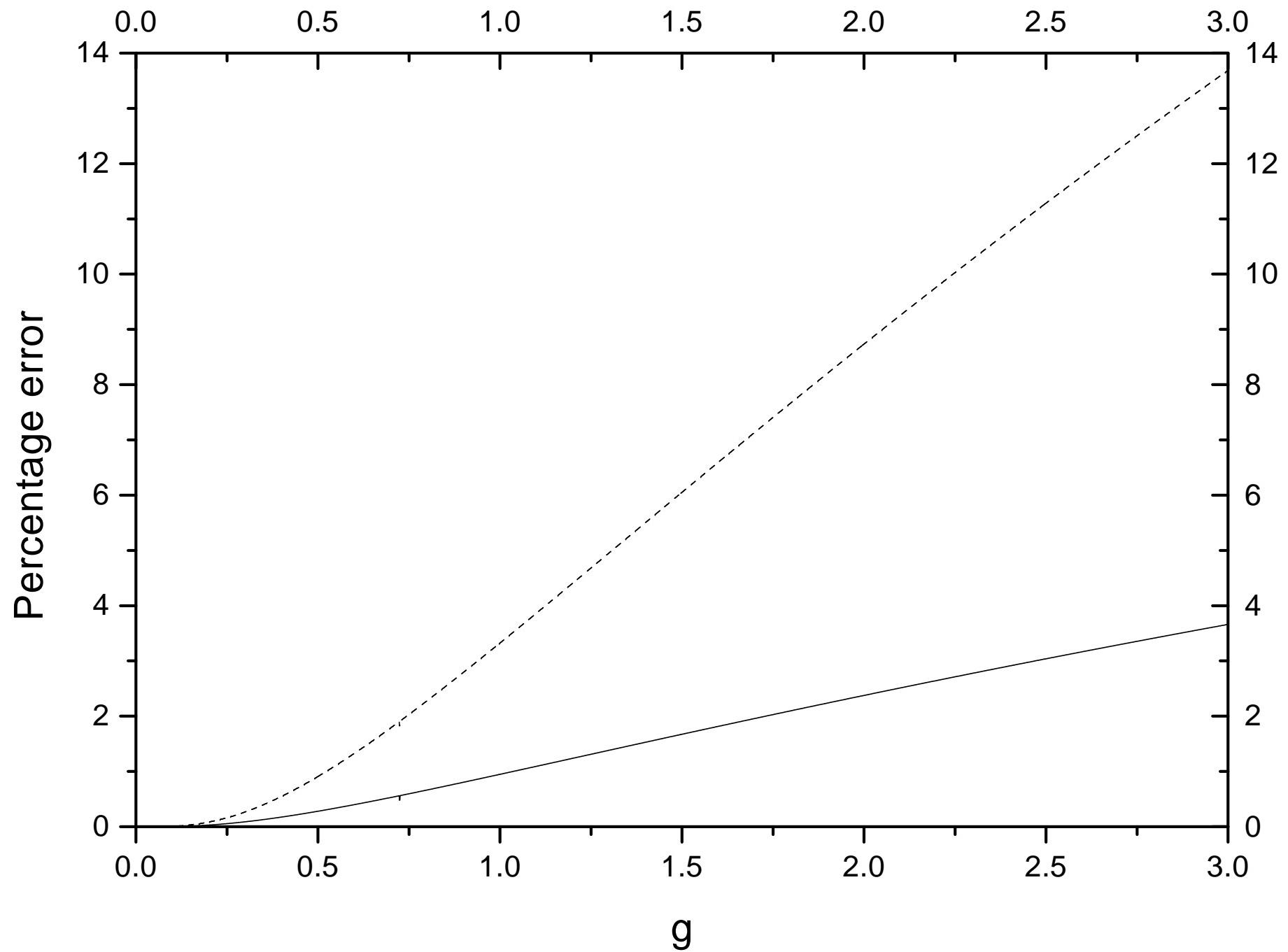


Fig. 7

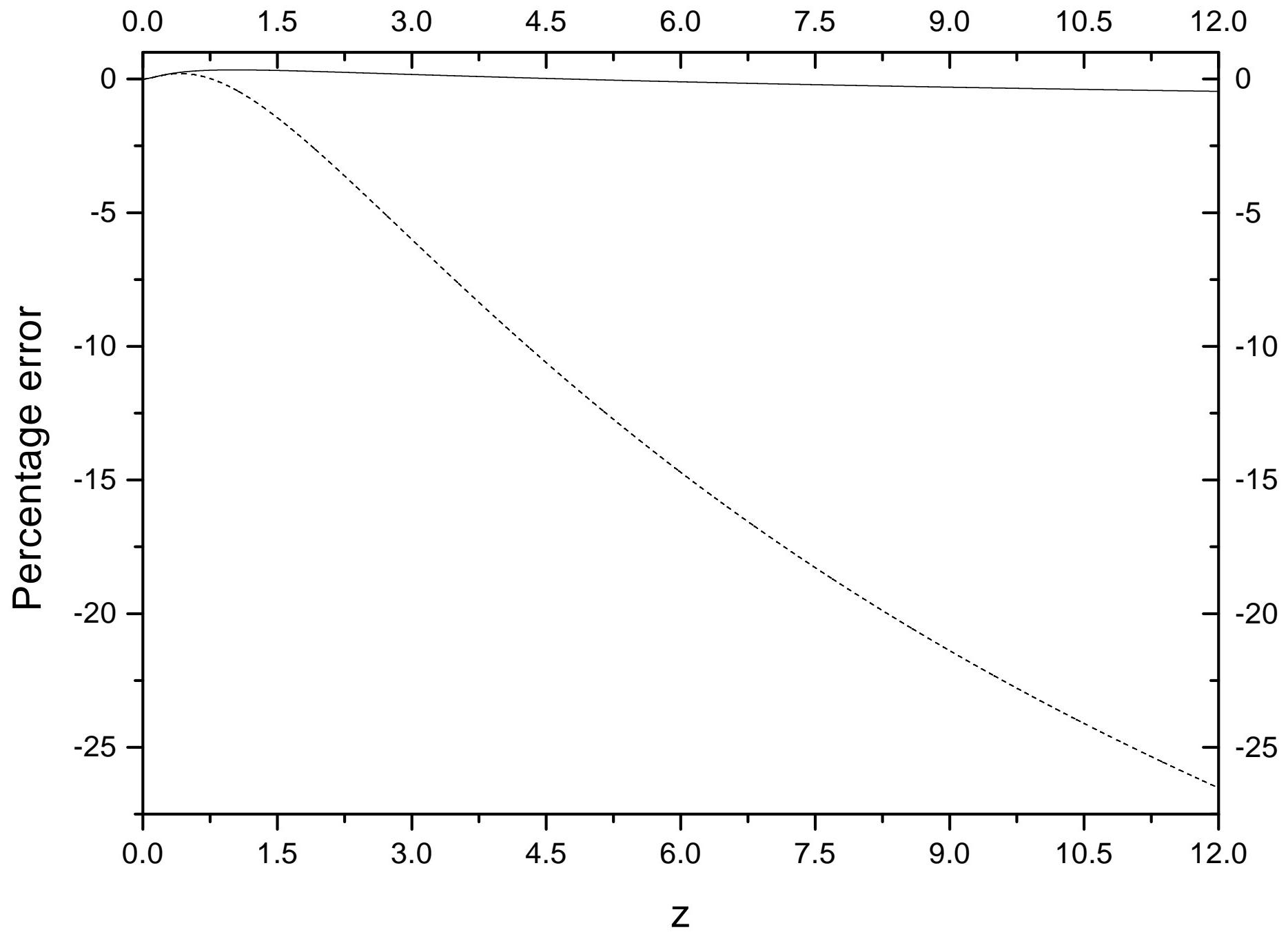


Fig. 8

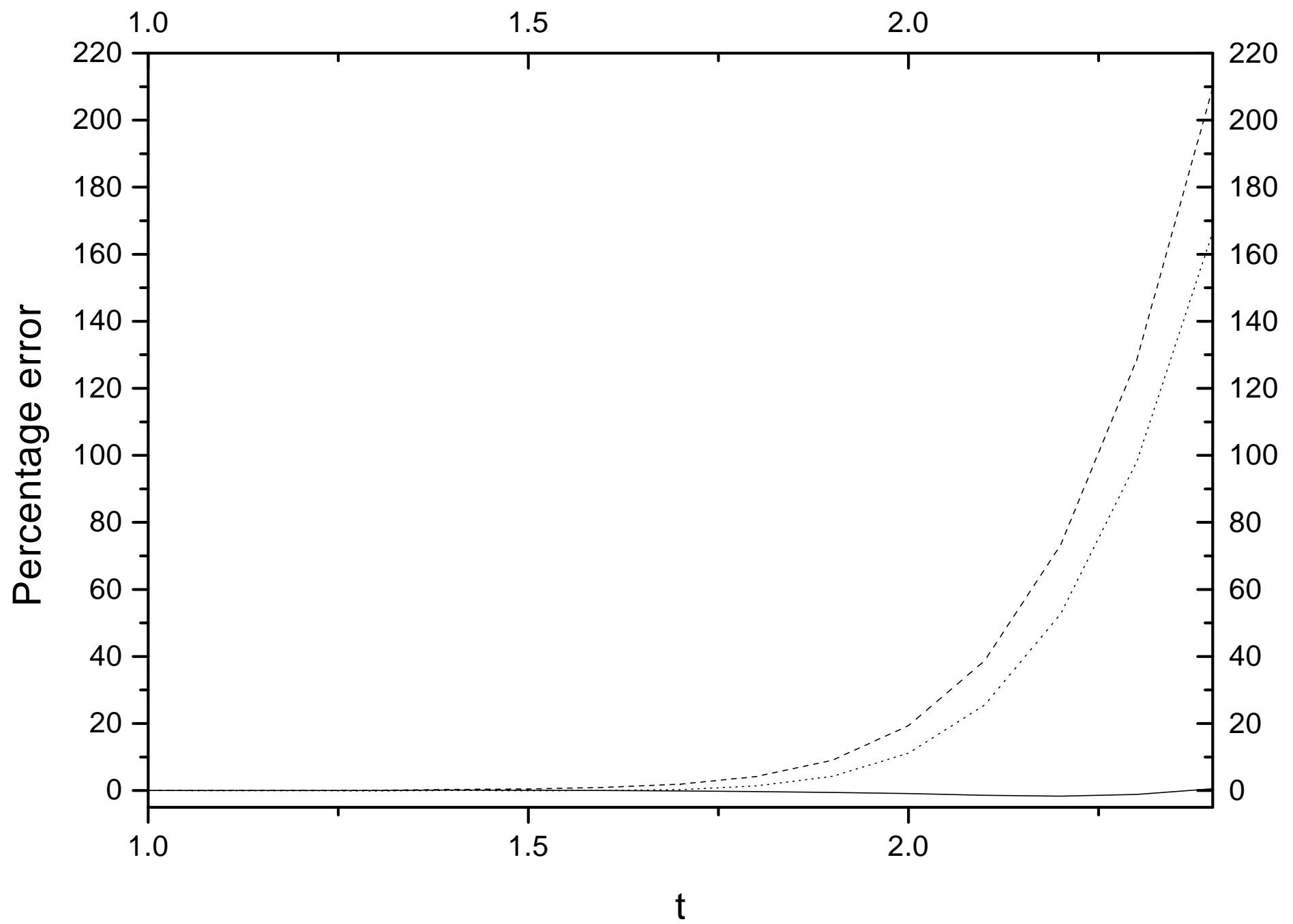


Fig. 9

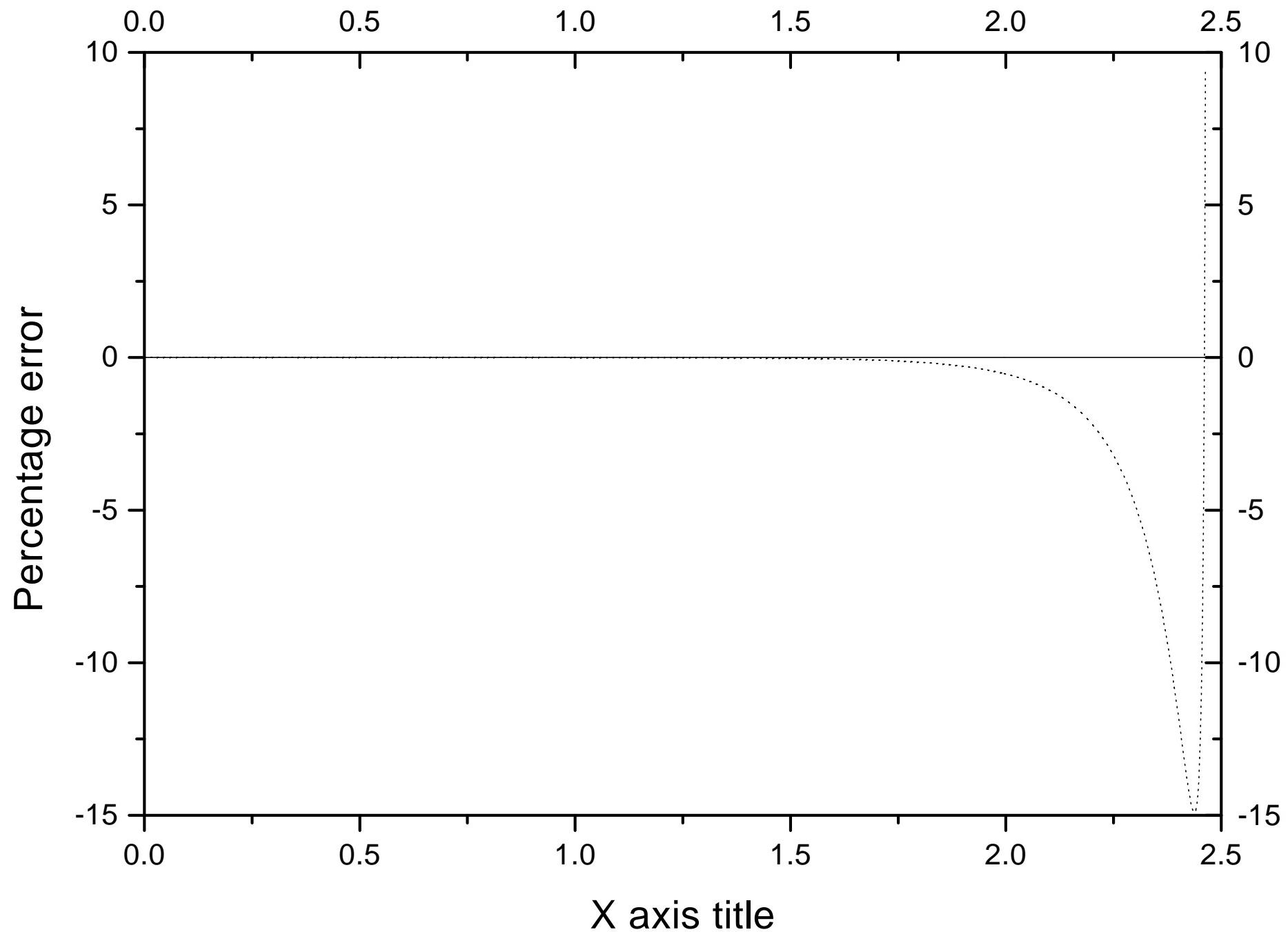


Fig. 10

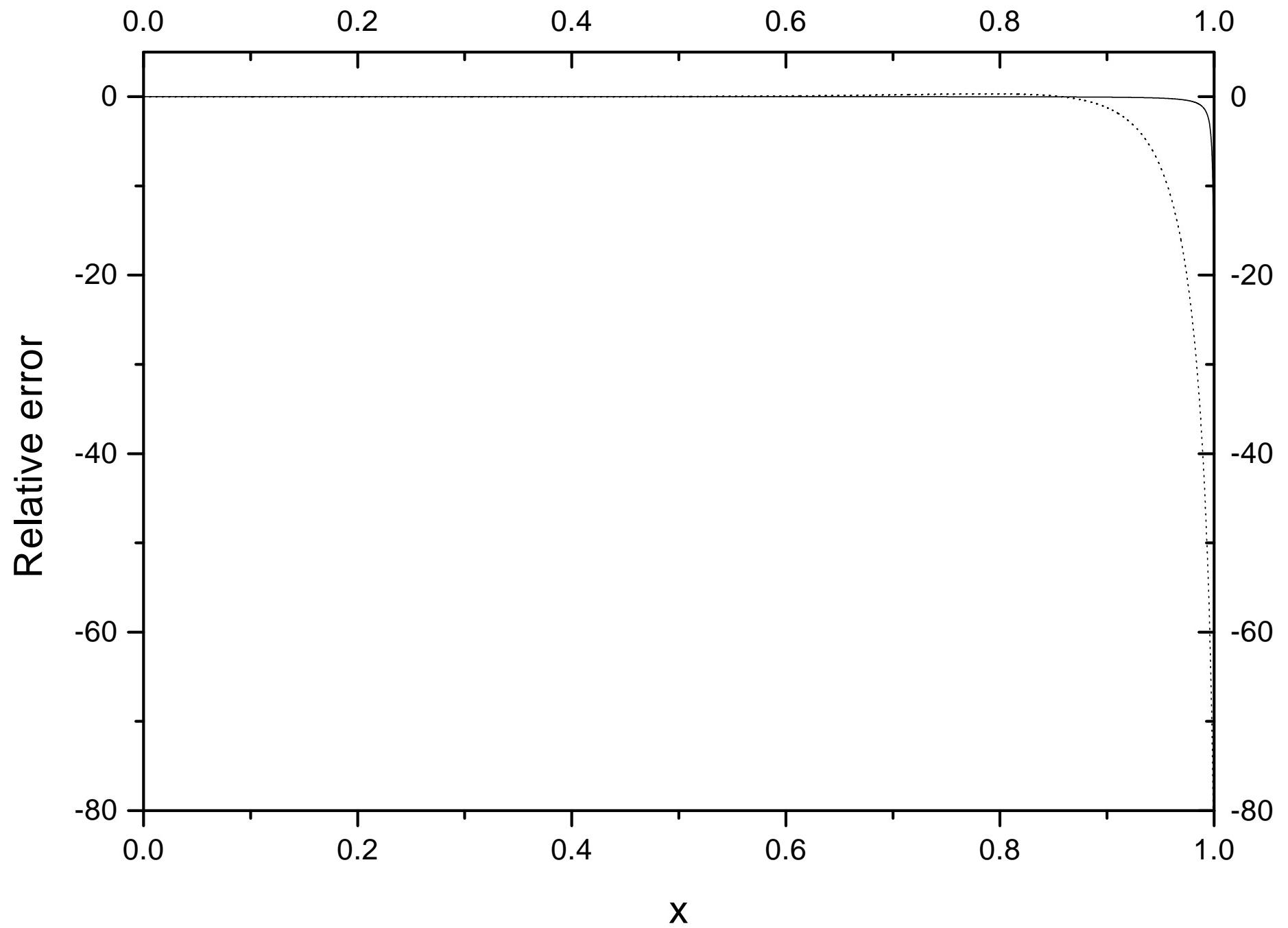


Fig. 11

